3.4 From Fourier-frequency to Carrier-frequency Domain

Often, when dealing with laser or microwave frequency standards, one is interested in the power spectrum of the oscillator in the carrier frequency domain. An ideal oscillator operating at the frequency ν_0 would consist of a delta function at ν_0 in the carrier frequency domain. For a real oscillator perturbed by noise processes the power is spread over a frequency range around the centre frequency ν_0. The power spectrum can be measured by different methods. As a first method consider a bandpass filter whose centre frequency is tuned over a frequency range in the vicinity of the centre frequency of the oscillator. The power spectrum of the oscillator is directly related to the power transmitted through the filter measured as a function of the frequency setting of the filter. In the optical domain, a tuneable Fabry–Pérot interferometer (Section 4.3.1) is often chosen as a filter to sweep across laser lines. Another possibility of measuring the power spectrum in the carrier frequency domain is to feed the signal from the oscillator simultaneously to a parallel filter bank. The parallel filter bank can also be simulated by a fast Fourier transform of a digitised and numerically filtered signal. It has to be pointed out, however, that the concept of a power spectrum with a well defined form and linewidth is in general not applicable to all noise processes. As an example, consider a power spectral density with large $1/f$ contribution. For long observation times corresponding to low Fourier frequencies the central frequency may drift away and, hence, there is no unique "linewidth" as the measured width of the power spectrum will depend on the observation time.

With this note of caution in mind, we show in this section how the shape of the emission line in the carrier frequency domain can be determined from a particular noise spectral density, e.g., $S_\nu(\nu)$ determined in the Fourier domain. The power spectrum of the electric field $S_E(\nu)$ can be evaluated by following [35–37]. In analogy to (3.27) and (3.28) one defines the two-sided power spectral density as the Fourier transform

$$S_E(\nu) = \int_{-\infty}^{\infty} \exp(-i2\pi\nu t)\, R_E(\tau) d\tau \tag{3.53}$$

of the autocorrelation function

$$R_E(\tau) = \big\langle E(t+\tau)E^*(t)\big\rangle \tag{3.54}$$

of the electric field $E(t)$. For a complex representation of the electric field of the electromagnetic wave with negligible amplitude fluctuations and real amplitude E_0

$$E(t) = E_0 \exp i[2\pi\nu_0 t + \phi(t)] \tag{3.55}$$

the autocorrelation function becomes

$$R_E(\tau) = E_0^2 \exp[i2\pi\nu_0\tau]\big\langle \exp i[\phi(t+\tau) - \phi(\tau)]\big\rangle. \tag{3.56}$$

Now, $\langle \exp i[\phi(t+\tau) - \phi(\tau)]\rangle$ has to be expressed in terms of the spectral density of phase fluctuations $S_\phi(f)$. To begin with, one assumes that the noise process is ergodic, i.e., that the temporal average is identical to the corresponding ensemble average

$$\overline{\exp[i\Phi(t,\tau)]} = \big\langle \exp[i\Phi(t,\tau)]\big\rangle = \int_{-\infty}^{\infty} p(\Phi)\exp(i\Phi)d\Phi \tag{3.57}$$

where

$$\Phi(t,\tau) \equiv \phi(t+\tau) - \phi(\tau) \tag{3.58}$$

is the phase accumulated during the interval τ. The right-hand side of (3.57) uses the usual definition of the expectation value of the quantity $\exp[i\Phi(t,\tau)]$ if the probability density $p(\Phi)$ is known. For a large number of uncorrelated phase-shifting events the central limit theorem allows one to use the Gaussian probability density

$$p(\Phi) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{\Phi^2}{2\sigma^2}\right) \tag{3.59}$$

with the classical variance σ^2. As $p(\Phi)$ is an even function, only the real (cosine) part of the complex exponential of (3.57) survives. (3.57) is evaluated by using (3.59) and $\int_{-\infty}^{\infty} \exp(-a^2x^2)\cos x dx = \sqrt{\pi}/a \exp(1/4a^2)$ leading to

$$\left\langle \exp[i\Phi(t,\tau)] \right\rangle = \exp\left(-\frac{\sigma^2}{2}\right). \tag{3.60}$$

According to (3.11) with vanishing mean value $\langle\Phi\rangle = 0$ and (3.58)

$$\begin{aligned} \sigma^2(\Phi) &= \left\langle \Phi^2 \right\rangle = \left\langle [\phi(t+\tau) - \phi(\tau)]^2 \right\rangle \\ &= \left\langle [\phi(t+\tau)]^2 \right\rangle - 2\left\langle [\phi(t+\tau)\phi(\tau)] \right\rangle + \left\langle [\phi(\tau)]^2 \right\rangle. \end{aligned} \tag{3.61}$$

Using (3.54) and (3.32) one finds

$$\left\langle [\phi(t+\tau)\phi(\tau)] \right\rangle = \int_0^\infty S_\phi(f)\cos(2\pi f\tau) df = R_\phi(\tau) \tag{3.62}$$

$$\left\langle [\phi(t+\tau)]^2 \right\rangle = \left\langle [\phi(\tau)]^2 \right\rangle = \int_0^\infty S_\phi(f) df = R_\phi(0). \tag{3.63}$$

Insertion of (3.62) and (3.63) into (3.61) leads to

$$\sigma^2 = 2\int_0^\infty S_\phi(f)\left[1 - \cos 2\pi f\tau)\right] df \tag{3.64}$$

which can be used to derive the autocorrelation function from (3.56)

$$R_E(\tau) = E_0^2 \exp(i2\pi\nu_0\tau) \exp\left(-\int_0^\infty S_\phi(f)\left[1 - \cos 2\pi f\tau)\right] df\right). \tag{3.65}$$

From (3.53) and (3.65) the power spectral density in the carrier frequency domain

$$S_E(\nu - \nu_0) = E_0^2 \int_{-\infty}^{\infty} \exp -[i2\pi(\nu-\nu_0)\tau] \exp\left(-\int_0^\infty S_\phi(f)\left[1 - \cos 2\pi f\tau)\right] df\right) d\tau \tag{3.66}$$

can be derived for a given phase noise spectral density $S_\phi(f)$ (see (3.37)) provided that the integral in brackets in (3.66) converges.

3.4.1 Power Spectrum of a Source with White Frequency Noise

We now consider a source whose power spectral density in the Fourier-frequency domain can be represented as white (frequency independent) frequency noise S_ν^0 (see Table 3.1). Consequently,

$$S_\phi(f) = \frac{S_\nu^0}{f^2} = \frac{\nu_0^2 h_0}{f^2} \tag{3.67}$$

holds and the integral in the exponential of (3.66) can be solved analytically using $\int_0^\infty [1 - \cos(bx)]/x^2 dx = \pi|b|/2$ leading to

$$\begin{aligned} S_E(\nu - \nu_0) &= E_0^2 \int_{-\infty}^{\infty} \exp -[i2\pi(\nu - \nu_0)\tau] \exp\left(-\pi^2 h_0 \nu_0^2 |\tau|\right) d\tau \\ &= 2E_0^2 \int_0^{\infty} \exp -\tau \left[i2\pi(\nu - \nu_0) + \pi^2 h_0 \nu_0^2\right] d\tau. \end{aligned} \tag{3.68}$$

Solving the integral (3.68) and keeping the real part leads to the power spectral density of

$$S_E(\nu - \nu_0) = 2E_0^2 \frac{h_0 \pi^2 \nu_0^2}{h_0^2 \pi^4 \nu_0^4 + 4\pi^2(\nu - \nu_0)^2} = 2E_0^2 \frac{\gamma/2}{(\gamma/2)^2 + 4\pi^2(\nu - \nu_0)^2} \tag{3.69}$$

with $\gamma \equiv 2h_0\pi^2\nu_0^2 = 2\pi(\pi h_0 \nu_0^2) = 2\pi(\pi S_\nu^0)$. Hence, the power spectral density of frequency fluctuations in the carrier-frequency domain of an oscillator with white frequency noise S_ν^0 in the Fourier-frequency domain, is a Lorentzian whose full width at half maximum is given by

$$\Delta\nu_{\mathrm{FWHM}} = \pi S_\nu^0. \tag{3.70}$$

Similarly, other types of phase noise spectral densities can be calculated accordingly. Godone and Levi have furthermore treated the case of white phase noise and flicker phase noise [38].

3.4.2 Spectrum of a Diode Laser

As an example of white frequency noise, consider the frequency fluctuations in a laser resulting from the spontaneous emission of photons [39]. They lead to the so-called Schawlow–Townes linewidth

$$\Delta\nu_{\mathrm{QNL}} = \frac{2\pi h\nu_0 (\Delta\nu_{1/2})^2 \mu}{P}. \tag{3.71}$$

where $h\nu_0$ is the photon energy, $\Delta\nu_{1/2}$ is the full width at half maximum of the passive laser resonator, $\mu \equiv N_2/(N_2 - N_1)$ is a parameter describing the population inversion in the laser medium, and P is the output power of the laser. This quantum-noise limited power spectral density (which is enhanced for laser diodes by Henry's linewidth enhancement factor; see (9.37)) can be found in the measured spectral noise of a solitary diode laser (Fig. 3.10) at Fourier frequencies above a corner frequency of about 80 kHz. At frequencies below the

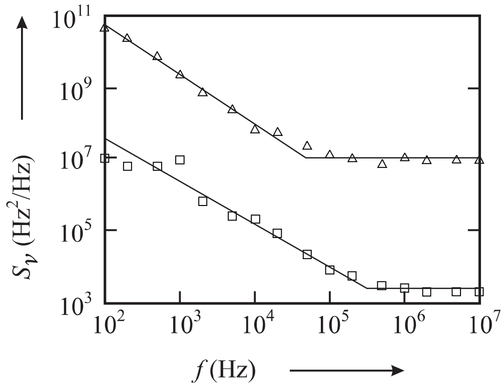

Figure 3.10: Measured power spectral densities of frequency fluctuations versus Fourier frequency f of a diode laser without optical feedback (triangles) and with optical feedback from a grating (squares) after [40] with permission.

corner frequency the power spectral density increases with a power law of roughly $1/f$. The white frequency noise regime is also visible above the corner frequency f_c of about 200 kHz if the cavity of the diode laser is extended (Section 9.3.2.5) but $S_\nu(f)$ is reduced by about 33 dB according to the reduced linewidth $\Delta\nu_{1/2}$ (see (3.71)).

As the $1/f$-like behaviour often results from technical noise which is present in any oscillator to some degree it is interesting to investigate the validity of (3.69). O'Mahony and Henning [41] have investigated the effect of low frequency (1/f) carrier noise on the linewidth of a semiconductor laser. From their findings Koch [40] gives a criterion that allows one to obtain information about the lineshape from the positions of the corner frequencies f_c as follows

$$S_\nu(f_c)/f_c \gg 1 :\Rightarrow \quad \text{Lorentzian lineshape} \tag{3.72}$$
$$S_\nu(f_c)/f_c \ll 1 :\Rightarrow \quad \text{Gaussian lineshape.} \tag{3.73}$$

We apply these criteria to the power spectral density of frequency noise displayed in Fig. 3.10 where one finds, for the solitary laser diode (triangles), $S_\nu(f_c)/f_c > 100$ and, hence, criterion (3.72) applies. With (3.70) one expects a Lorentzian profile of about 5 MHz linewidth. From the power spectral density of frequency fluctuations (squares in Fig. 3.10) of another diode laser with extended cavity (Section 9.3.2.5) one finds $S_\nu(f_c)/f_c \approx 10^{-2}$ and hence expects a Gaussian lineshape according to criterion (3.73). The origin of the Gaussian lineshape can be thought of as resulting from a small Lorentzian line whose width is given by (3.70) which statistically wanders around a central frequency. The width of the Gaussian depends on the time T of averaging, as the measurement time T also defines the lowest measurable Fourier frequency $1/T$. For a true $1/f$ behaviour of S_ν the linewidth would be infinite as $\int_{1/T}^{\infty} S_\nu(f)df = \infty$ holds (see (3.66)). Experimentally, however, one always finds a finite linewidth resulting from the finite measurement time T with the low-frequency cut off $1/T$.

The mean frequency excursion $\Delta\nu_{\mathrm{rms}}$ (linewidth) can be computed as

$$\Delta\nu_{\mathrm{rms}} = \sqrt{\int_{1/T}^{f_c} S_\nu(f)df} \tag{3.74}$$

from (3.39). In the case of the laser with optical feedback in an extended cavity arrangement (squares in Fig. 3.10) one derives a FWHM of the Gaussian of about 120 kHz for a measurement time of 10 ms.

3.4.3 Low-noise Spectrum of a Source with White Phase Noise

With the help of (3.62) and (3.63) we can write (3.66) as

$$S_E(\nu-\nu_0) = E_0^2 \int_{-\infty}^{\infty} \exp\left[-R_\phi(0)\right] \exp\left[R_\phi(\tau)\right] \exp[-i2\pi(\nu-\nu_0)\tau]d\tau. \tag{3.75}$$

For very low phase fluctuations, i.e., for $\int_0^\infty S_\phi(f)df \ll 1$ it is justified to expand the first two exponential functions in (3.75) and to keep only the first terms as

$$S_E(\nu-\nu_0) \approx E_0^2 \int_{-\infty}^{\infty} [1 - R_\phi(0) + R_\phi(\tau)] \exp[-i2\pi(\nu-\nu_0)\tau]d\tau. \tag{3.76}$$

Using the definition of Dirac's delta function $\delta(\nu-\nu_0)$ (see (2.23)) and the Wiener–Khintchine relation (3.28) one finds

$$S_E(\nu-\nu_0) \approx E_0^2 \left[1 - R_\phi(0)\right] \delta(\nu-\nu_0) + E_0^2 S_\phi^{2-\mathrm{sided}}(\nu-\nu_0). \tag{3.77}$$

Hence, the spectrum in the carrier frequency domain comprises a carrier (delta function) at $\nu = \nu_0$ and two symmetric sidebands with the level of the phase noise spectral density S_ϕ at $f = |\nu-\nu_0|$.

Often commercial oscillators are specified by the measure of the so-called spectral purity $\mathcal{L}(f)$, i.e., the noise found on each side of the carrier when the signal of an oscillator is measured directly with a spectrum analyser [1]

$$\mathcal{L}(f) \equiv \frac{S_\Phi^{2-\mathrm{sided}}(\nu-\nu_0)}{1/2E_0^2}. \tag{3.78}$$

Here it is assumed that the amplitude noise is negligible as compared to the phase noise. Then the spectral purity represents all phase noise for all Fourier frequencies except for the origin, i.e., the delta function of (3.77).